

Correspondence

Resonant Frequency Function of Thickness-Shear Vibrations of Rectangular Crystal Plates

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Abstract—The resonant frequencies of thickness-shear vibrations of quartz crystal plates in rectangular and circular shapes are always required in the design and manufacturing of quartz crystal resonators. As the size of quartz crystal resonators shrinks, for rectangular plates we must consider effects of both length and width for the precise calculation of resonant frequency. Starting from the three-dimensional equations of wave propagation in finite crystal plates and the general expression of vibration modes, we obtained the relations between frequency and wavenumbers. By satisfying the major boundary conditions of the dominant thickness-shear mode, three wavenumber solutions are obtained and the frequency equation is constructed. It is shown the resonant frequency of thickness-shear mode is a second-order polynomial of aspect ratios. This conforms to known results in the simplest form and is applicable to further analytical and experimental studies of the frequency equation of quartz crystal resonators.

I. INTRODUCTION

As the core issue of a quartz crystal resonator, vibration analysis of a crystal plate, usually circular or rectangular, is always required for the essential information regarding properties such as the vibration frequency and mode coupling. Such characteristics are generally required either as properties of the resonator or in the determination of other properties for an electrical circuit element in applications. In either case, the analysis of vibrations of a crystal plate in the thickness-shear mode, which is one of important functioning modes of quartz crystal resonators, must be done accurately to determine design parameters which will reduce the coupling with spurious modes and enhance the resonance through the quality factor. This has long been common knowledge in the study of quartz crystal resonators, and efforts have been made to analyze vibrations of rectangular quartz crystal plates. To overcome the difficulties in dealing with three-dimensional equations of wave propagation in elastic

solids, Mindlin established the two-dimensional equations for the vibrations of elastic plates [1]. The equations for thickness-shear vibrations of plates, widely known as the Mindlin plate equations, have been utilized for the calculation of vibration frequencies of rectangular plates by Mindlin and others [2]–[6]. It has been clear that assumptions, approximations, and corrections have been made in these studies for the considerations of complications like the presence of electrodes and beveling (thickness variation) of crystal plates in resonators [7], [8]. Although accurate and reliable results have been obtained through the efforts of Mindlin and his collaborators, further studies have focused on improved analysis, including the closed-form solutions based on an approximation of boundary conditions [9]. For accurate solutions of the vibration frequency and mode shapes of thickness-shear vibrations of rectangular plates, the finite element method based on the Mindlin plate equations has been used with success [10], [11]. Of course, with additional cost in computing, commercial general purpose finite element analysis software applications have been used in the analysis of thickness-shear vibrations of quartz crystal resonators [12]–[15]. All of these methods and approaches, spanning half a century of research with the aid of technology of time, have been shown that emerging problems and practical needs must be fully addressed, and further studies and solutions are critical in the design and improvement of newer generations of quartz crystal resonators in the wake of fast-shrinking sizes and rapidly increasing frequency. Consequently, relatively amplified effects of complications such as electrodes and mountings structures require the accurate prediction of vibrations for the optimal design to achieve better performance. In other words, precise analysis of thickness-shear vibrations of crystal plates is even more important now.

As part of the procedure to improve and complement methods and results on thickness-shear vibrations of crystal plates, we start, again, with the three-dimensional theory of vibrations of elastic solids and known knowledge like the dispersion relations [1], [16]–[18]. Then, through the examination of major vibration modes and boundary conditions, a relationship between the vibration frequency and wavenumbers is established in a manner similar to that demonstrated by earlier researchers [3], [19], [20]. However, by retaining the smaller terms, in comparison to those of the dominant thickness-shear mode in the frequency functions, known frequency expressions are complemented with the linear terms of the length and width of a plate. This small change will keep the known trend of the dominant quadratic terms of the length and width and improve the long-term and overall accuracy of the thickness-shear resonant frequency, which is needed for the design and manufacturing of quartz crystal resonators for fast determination of device structural parameters.

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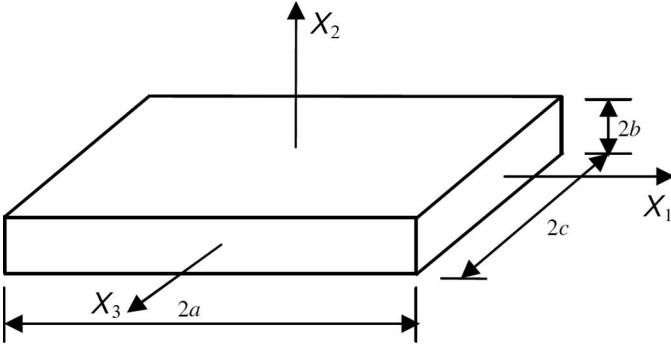


Fig. 1 A rectangular plate with coordinate system.

Our results, as expected by design engineers, will meet the practical needs and provide foundation for the combination of analytical method in the design and measurement from actual production.

II. THREE-DIMENSIONAL EQUATIONS OF VIBRATIONS OF PLATES

Vibrations of elastic solids are governed by the three-dimensional equations of elasticity, which are applicable to plates with simplifications [1]. For a rectangular plate with coordinates as shown in Fig. 1, we have the following equations of motion in abbreviated notation [1]:

$$\begin{aligned} \frac{\partial T_1}{\partial x_1} + \frac{\partial T_6}{\partial x_2} + \frac{\partial T_5}{\partial x_3} &= \rho \frac{\partial^2 u_1}{\partial t^2}, \\ \frac{\partial T_6}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \frac{\partial T_4}{\partial x_3} &= \rho \frac{\partial^2 u_2}{\partial t^2}, \\ \frac{\partial T_5}{\partial x_1} + \frac{\partial T_4}{\partial x_2} + \frac{\partial T_3}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2}, \end{aligned} \quad (1)$$

where T_p ($p = 1, 2, 3, 4, 5, 6$), x_i ($i = 1, 2, 3$), u_i ($i = 1, 2, 3$), and t are the stresses, coordinates, displacements, and time, respectively.

For AT-cut plates of quartz crystal, the constitutive relations are

$$\begin{aligned} T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4, \\ T_2 &= c_{21}S_1 + c_{22}S_2 + c_{23}S_3 + c_{24}S_4, \\ T_3 &= c_{31}S_1 + c_{32}S_2 + c_{33}S_3 + c_{34}S_4, \\ T_4 &= c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4, \\ T_5 &= c_{55}S_5 + c_{56}S_6, \\ T_6 &= c_{65}S_5 + c_{66}S_6, \end{aligned} \quad (2)$$

where c_{pq} ($p, q = 1, 2, 3, 4, 5, 6$) are the elastic constants, and the strain components are

$$\begin{aligned} S_1 &= u_{1,1}, & S_2 &= u_{2,2}, \\ S_3 &= u_{3,3}, & S_4 &= u_{3,2} + u_{2,3}, \\ S_5 &= u_{3,1} + u_{1,3}, & S_6 &= u_{2,1} + u_{1,2}. \end{aligned} \quad (3)$$

By substituting (3) into (2), we have the following stress-displacement relations:

$$\begin{aligned} T_1 &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}u_{3,2} + c_{14}u_{2,3}, \\ T_2 &= c_{21}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}u_{3,2} + c_{24}u_{2,3}, \\ T_3 &= c_{31}u_{1,1} + c_{32}u_{2,2} + c_{33}u_{3,3} + c_{34}u_{3,2} + c_{34}u_{2,3}, \\ T_4 &= c_{41}u_{1,1} + c_{42}u_{2,2} + c_{43}u_{3,3} + c_{44}u_{3,2} + c_{44}u_{2,3}, \\ T_5 &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{2,1} + u_{1,2}), \\ T_6 &= c_{65}(u_{3,1} + u_{1,3}) + c_{66}(u_{2,1} + u_{1,2}). \end{aligned} \quad (4)$$

Finally, by substituting (4) into (1), we have the displacement equations of motion:

$$\begin{aligned} c_{11}u_{1,11} + c_{12}u_{2,21} + c_{13}u_{3,31} + c_{14}(u_{3,21} + u_{2,31}) \\ + c_{55}(u_{3,13} + u_{1,33}) + c_{56}(u_{2,13} + u_{1,23}) \\ + c_{65}(u_{3,12} + u_{1,32}) + c_{66}(u_{2,12} + u_{1,22}) &= \rho \frac{\partial^2 u_1}{\partial t^2}, \\ c_{21}u_{1,12} + c_{22}u_{2,22} + c_{23}u_{3,32} + c_{24}(u_{3,22} + u_{2,32}) \\ + c_{41}u_{1,13} + c_{42}u_{2,23} + c_{43}u_{3,33} + c_{44}(u_{2,33} + u_{3,23}) \\ + c_{65}(u_{3,11} + u_{1,31}) + c_{66}(u_{2,11} + u_{1,21}) &= \rho \frac{\partial^2 u_2}{\partial t^2}, \\ c_{31}u_{1,13} + c_{32}u_{2,23} + c_{33}u_{3,33} + c_{34}(u_{2,33} + u_{3,23}) \\ + c_{41}u_{1,12} + c_{42}u_{2,22} + c_{43}u_{3,32} + c_{44}(u_{2,32} + u_{3,22}) \\ + c_{55}(u_{3,11} + u_{1,31}) + c_{56}(u_{2,11} + u_{1,21}) &= \rho \frac{\partial^2 u_3}{\partial t^2}. \end{aligned} \quad (5)$$

These equations are generally known, and we shall use them as the starting point for our study.

First, displacements of vibrations of a crystal plate shown in Fig. 1 can be denoted

$$u_j = A_j e^{i(\xi x_1 + \eta x_2 + \zeta x_3 - \omega t)}, \quad j = 1, 2, 3, \quad (6)$$

where A_j ($j = 1, 2, 3$), ξ , η , ζ , and ω are amplitudes, wave-number of the x_1 direction, wavenumber of the x_2 direction, wavenumber of the x_3 direction, and angular frequency, respectively.

For further simplification, we define the normalized variables (5) as

$$\Omega = \frac{\omega}{\frac{\pi}{2b} \sqrt{\frac{c_{66}}{\rho}}}, \quad X = \frac{\xi}{\frac{\pi}{2b}}, \quad Y = \frac{\eta}{\frac{\pi}{2b}}, \quad Z = \frac{\zeta}{\frac{\pi}{2b}}, \quad C_{pq} = \frac{c_{pq}}{c_{66}}. \quad (7)$$

With (6) and (7), we have the equations of motion in the form

$$\begin{aligned} [C_{11}X^2 + Y^2 + C_{55}Z^2 + (C_{65} + C_{56})YZ - \Omega^2]A_1 \\ + [(C_{12} + 1)XY + (C_{14} + C_{56})XZ]A_2 \\ + [(C_{13} + C_{55})XZ + (C_{14} + C_{65})XY]A_3 = 0, \\ [(1 + C_{21})XY + (C_{65} + C_{41})XZ]A_1 \\ + [X^2 + C_{22}Y^2 + C_{44}Z^2 + (C_{24} + C_{42})YZ - \Omega^2]A_2 \\ + [C_{65}X^2 + C_{24}Y^2 + C_{43}Z^2 + (C_{23} + C_{44})YZ]A_3 = 0, \\ [(C_{56} + C_{41})XY + (C_{55} + C_{31})XZ]A_1 \\ + [C_{56}X^2 + C_{42}Y^2 + C_{34}Z^2 + (C_{44} + C_{32})YZ]A_2 \\ + [C_{55}X^2 + C_{44}Y^2 + C_{33}Z^2 + (C_{34} + C_{43})YZ - \Omega^2]A_3 \\ = 0. \end{aligned} \quad (8)$$

These equations show the relationship of the vibration frequency, amplitudes, wavenumbers, and elastic constants. This is an important step in establishing the frequency equation for the vibration analysis. The dispersion relation is obtained by setting the determinant of coefficient matrix of amplitudes in (8) to vanish [1], [6], [7], [19], [20], but we shall skip this part and take a different approach.

III. FREQUENCY FUNCTION OF THE THICKNESS-SHEAR VIBRATIONS OF PLATES

To determine the vibration frequency of a plate, we need to make sure that the displacements we assumed in (6) satisfy the boundary conditions of a plate shown in Fig. 1. This, of course, is a difficult problem and three-dimensional equations presented in the previous section and even the two-dimensional Mindlin plate equations cannot be solved with this approach. In turn, to establish the relationship between the vibration frequency and sizes of plate, some approximations are needed through the examination of the boundary conditions corresponding to the thickness-shear vibration mode.

First of all, for the plate in Fig. 1, the precise boundary conditions for major traction-free faces of free vibrations are

$$\begin{aligned} T_2 = T_4 = T_6 = 0, \quad x_2 = \pm b, \\ T_1 = T_5 = T_6 = 0, \quad x_1 = \pm a, \\ T_3 = T_4 = T_5 = 0, \quad x_3 = \pm c, \end{aligned} \quad (9)$$

where a , b , and c are the half-length, thickness, and width of the plate in Fig. 1. The problem is now to find displacements which will also satisfy the traction-free stress boundary conditions in (12).

As we stated from the beginning, most quartz crystal resonators vibrate at the thickness-shear mode of the crystal plate. For this reason, we shall need to find the displacements of the thickness-shear mode satisfying conditions in (9). Such a displacement solution satisfying all boundary conditions does not exist, and approximations must be made to find the dominant conditions to be satisfied by the thickness-shear mode. This can be done by selecting the thickness-shear vibrations matching the symmetric feature on the plane and antisymmetric regarding the plate thickness.

Assuming the thickness-shear vibration mode taking the form of [1], [16]–[18]

$$u_1 = A_1 \cos \xi x_1 \sin \eta x_2 \cos \zeta x_3 e^{i\omega t}, \quad (10)$$

we have displacement representing the thickness-shear mode of quartz crystal resonators. We now need to examine the boundary conditions to relax the requirements with simplifications.

First, in the two major faces denoted by $x_2 = \pm b$, for the traction-free boundary conditions given in (9), we must have the stress boundary conditions

$$T_2 = T_4 = T_6 = 0, \quad x_2 = \pm b. \quad (11)$$

Because the dominant displacement is u_1 , the corresponding dominant stress boundary condition will be $T_6 = 0$ with the negligence of other displacement and stress components. With the constitutive relations in (4), this boundary condition implies

$$\cos \eta b = 0, \quad (12)$$

or

$$\eta = \frac{n\pi}{2b}, \quad n = 1, 3, 5, \dots, \quad (13)$$

for the antisymmetric thickness deformation. Through the normalization in (7), we have

$$Y = \frac{\eta}{\frac{\pi}{2b}} = n. \quad (14)$$

Now we turn to the two faces at the ends of plate in the length direction. The traction-free boundary conditions from (9) are

$$T_1 = T_5 = T_6 = 0, \quad x_1 = \pm a. \quad (15)$$

Again, because only the displacement u_1 is considered, the dominant boundary condition will now be $T_6 = 0$ [17]. With the constitutive relations in (4), we have

$$\cos \xi a = 0, \quad (16)$$

or the wavenumbers are

$$\xi = \frac{m\pi}{2a}, \quad m = 1, 3, 5, \dots \quad (17)$$

Again, through normalization defined in (10) we have

$$X = \frac{\xi}{\frac{\pi}{2a}} = \frac{mb}{a}. \quad (18)$$

Third, on the two faces at the ends of width $x_3 = \pm c$, the complete set of traction-free boundary conditions from (9) is

$$T_3 = T_4 = T_5 = 0, \quad x_3 = \pm c. \quad (19)$$

Given the known displacement u_1 , only the condition $T_5 = 0$ is required [17], or

$$\sin \zeta c = 0, \quad (20)$$

which gives

$$\zeta = \frac{l\pi}{2c}, \quad l = 0, 2, 4, \dots \quad (21)$$

Then the normalized wavenumber is

$$Z = \frac{\zeta}{\frac{\pi}{2b}} = \frac{lb}{c}. \quad (22)$$

Because the first equation of (8), representing the thickness-shear vibration mode, can be rewritten as

$$\begin{aligned} \Omega^2 &= C_{11}X^2 + Y^2 + C_{55}Z^2 + (C_{65} + C_{56})YZ \\ &+ \frac{A_2}{A_1}[(C_{12} + 1)XY + (C_{14} + C_{56})XZ] \\ &+ \frac{A_3}{A_1}[(C_{13} + C_{55})XZ + (C_{14} + C_{65})XY], \end{aligned} \quad (23)$$

by substituting (14), (18), and (22) into (23), we have the vibration frequency in the form

$$\begin{aligned} \Omega^2 &= n^2 + \left[\frac{A_2}{A_1}(C_{12} + 1) + \frac{A_3}{A_1}(C_{14} + C_{65}) \right] \frac{mb}{a} + C_{11} \left(\frac{mb}{a} \right)^2 \\ &+ \left[\frac{A_2}{A_1}(C_{14} + C_{56}) + \frac{A_3}{A_1}(C_{13} + C_{55}) \right] \frac{mlb^2}{ac} \\ &+ (C_{65} + C_{56}) \frac{lb}{c} + C_{55} \left(\frac{lb}{c} \right)^2. \end{aligned} \quad (24)$$

We denote

$$\begin{aligned} H &= \left[\frac{A_2}{A_1}(C_{12} + 1) + \frac{A_3}{A_1}(C_{14} + C_{65}) \right] \frac{mb}{a} + C_{11} \left(\frac{mb}{a} \right)^2 \\ &+ \left[\frac{A_2}{A_1}(C_{14} + C_{56}) + \frac{A_3}{A_1}(C_{13} + C_{55}) \right] \frac{mlb^2}{ac} \\ &+ (C_{65} + C_{56}) \frac{lb}{c} + C_{55} \left(\frac{lb}{c} \right)^2, \end{aligned} \quad (25)$$

with normalized elastic constants for AT-cut quartz crystal as [21]

$$\begin{aligned} C_{11} &= 2.99, & C_{12} &= -0.2845, & C_{13} &= 0.9359, \\ C_{14} &= -0.1262, & C_{55} &= 2.3719, & C_{56} &= C_{65} = 0.0873. \end{aligned}$$

Because the thickness-shear vibration mode is dominant, at the resonance, we must have $A_1 \gg A_2$ and $A_1 \gg A_3$; for small b/a and b/c we always have $|H| \ll 1$. Accordingly, we can rewrite (24) as

$$\Omega = \sqrt{n^2 + H} \approx n + \frac{1}{2n}H. \quad (26)$$

Finally, the resonant frequency of the thickness-shear vibrations of a rectangular plate can be approximated in terms of aspect ratios and amplitude ratios:

$$\begin{aligned} \Omega &= n + \left[\frac{A_2}{A_1}(C_{12} + 1) + \frac{A_3}{A_1}(C_{14} + C_{65}) \right] \frac{mb}{2na} + \frac{C_{11}}{2n} \left(\frac{mb}{a} \right)^2 \\ &+ 2 \left[\frac{A_2}{A_1}(C_{14} + C_{56}) + \frac{A_3}{A_1}(C_{13} + C_{55}) \right] \frac{mlb^2}{2nac} \\ &+ (C_{65} + C_{56}) \frac{lb}{2nc} + \frac{C_{55}}{2n} \left(\frac{lb}{c} \right)^2. \end{aligned} \quad (27)$$

An expression similar to (27) has been obtained by many researchers before for the estimation of thickness-shear vibrations, which is needed in the design and analysis of quartz crystal resonators [16]–[18]. Such a formula is also of great importance in the initial design stage for the selection of the sizes of crystal blanks. If we further write the frequency of fundamental thickness-shear vibrations with $m = n = 1$, $l = 0$, (27) will be reduced to

$$\Omega = B_0 + B_1 \frac{b}{a} + B_2 \left(\frac{b}{a} \right)^2, \quad (28)$$

where constants B_i ($i = 0, 1, 2$) include combinations of elastic constants and amplitude ratios. However, B_0 has been represented as a variable to accommodate possible slight deviation from the original value of 1 in (27). It is then clear that the accurate frequency is dependent upon the aspect ratios of plate, and coefficients can be determined through the known frequency and aspect ratios. This is important in practical applications because the vibration frequency of the thickness-shear mode can be easily measured for given crystal plates, and the coefficients can be determined as a result. With stable manufacturing process and accurate measurements, we can establish the frequency equations for practical usage in the design and manufacturing process. Actually, (28) is the best estimation of thickness-shear vibration frequency of crystal plates because the width effect is small and can be neglected from many earlier studies.

To verify the frequency relationship presented in (28), we use the known thickness-shear frequencies with different length-to-thickness ratios to determine the coefficients. Because the thickness-shear frequencies are calculated from the coupled thickness-shear and flexural vibrations with the truncated and simplified Mindlin plate equations without the consideration of the effect of width [6], the frequency equation in (28) has a much simpler form with only the linear and quadratic terms of length-to-thickness ratio. By selecting the thickness-shear resonant frequencies through the frequency spectra of the weakest coupling with the flexural mode, or taking the central points of the thickness-shear branch between two flexural branches [6], we can use known length-to-thickness ratios for the determination of three coefficients of the second-order polynomial from (28). The relation from such a regression is

$$\Omega = 0.99962 + 0.02715 \frac{b}{a} + 1.40963 \left(\frac{b}{a} \right)^2, \quad (29)$$

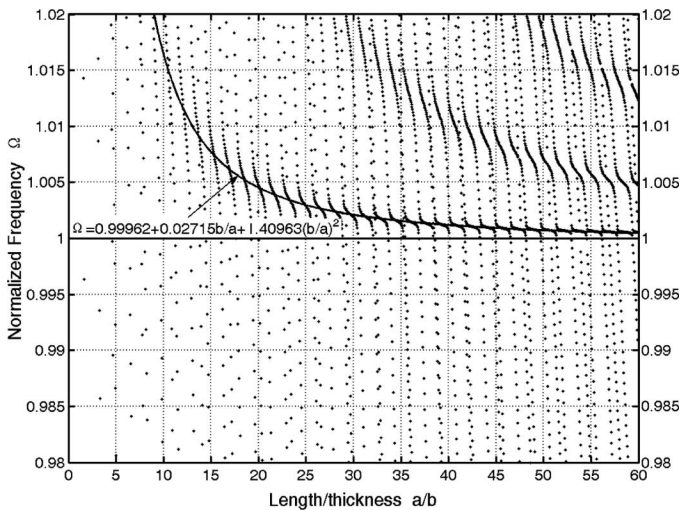


Fig. 2. Frequency spectra of coupled thickness-shear and flexural vibrations versus the length to thickness ratios and the thickness-shear resonant frequency from regression.

showing a weak link to the linear term. Earlier studies have neglected the linear terms [16]–[18], and our results have confirmed the justified approximation. Our regression also shows that the three indeterminate coefficients generally give better representation from our calculation. This suggests there are slight inaccuracies in thickness-shear vibration frequencies and inadequate numbers of sampling points for the regression. The frequency spectra and regression are shown in Fig. 2. The error of regression is not presented because we are more interested in the quadratic relation shown and proven in this study for future predictions based on actual measurements from quartz crystal blanks.

IV. CONCLUSIONS

Through approximations of the displacement of thickness-shear vibration mode and boundary conditions of a rectangular crystal plate, the resonant frequency of thickness-shear mode is given in terms of aspect ratios and unknown amplitude ratios. Because elastic constants and amplitude ratios can be treated as constants in coefficients, a second-order polynomial relationship between aspect ratios and frequency is established. This simple relation is an improvement upon earlier results, which have neglected the small contributions of the linear terms, making it slightly skewed when comparing with known resonant frequencies. This relation has practical applications in the design and manufacturing of crystal blanks because the frequency can be accurately predicted with accumulated data on sizes and frequencies through stable manufacturing process. As a demonstration of the accuracy of the relationship, a regression with known resonant frequencies of thickness-shear vibrations from the frequency spectra of coupled thickness-shear and flexural vibrations has produced a perfect match. Further regression of the measurement data with the relationship presented in this

study is being implemented for design and production applications.

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