Design of Quartz Crystal Resonators with an Analytical Procedure Based on the Mindlin Plate Theory

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Abstract—**By using the first-order Mindlin plate equations with the consideration of electrode and thermal effects, a theoretical model of AT-cut quartz crystal resonator is established. This one-dimensional model and analytical solutions can provide frequency, mode shapes, temperature effect, and electrical parameters like the capacitance ratios. The frequency-temperature relations and capacitance are also measured from samples with exact parameters for calculations. The mode coupling, mode conversion, and occurrences of spurious mode are examined through the measurement and compared with analytical results for mode identification. The parameters examined include that of the crystal blank, orientation, electrode, materials for fine tunings. Mindlin plate equations with couplings of thickness-shear, flexural, face-shear, extension, and other modes and electrical field have been considered for the frequency, displacements, and electrical potential solutions. Measurements are made for the frequency, which is also calculated with known equations with the consideration of electrodes to validate the equations and initial design.**

I. INTRODUCTION

The quartz crystal resonators for frequency control and detection applications have been subjected to the analysis of high frequency vibrations with objectives of the exact vibration frequency, mode coupling, and frequency stability under bias fields such as the temperature variation and acceleration for product improvements and optimization [1-7]. Earlier studies are more concentrated on the theoretical formulation and analytical solutions with simplified equations and simplest models to guide the improvement of actual designs based on experiences and promising prototypes. Such analyses have been made, in most times, with the tailored equations of vibrations of plates with specialized configurations like beveling and electrodes to focus on the thickness-shear (TSh) vibrations and effects of shape variations [5, 7]. The results of such analyses have been proven theoretically and experimentally and by the fact that many engineers have turned to the analytical theory and methods for the determination of design parameters and improvement of initial designs with success. These studies with focus on the quartz crystal resonators, a rare problem of wave propagation in piezoelectric solids, started from universities with pioneering activities on the core issues of high frequency vibrations of piezoelectric crystal plates and extensive connections with the frequency control industry with leading contributions from Columbia by Mindlin [1-2, 8-9], University of Tokyo by Koga [10], Rensselaer by Tiersten [3], Princeton by Lee [4-5, 11], Rutgers by Yong [6], and lately Nebraska–Lincoln by Yang [1, 7, 12] and others [7, 13-17, 18]. These studies are presented and published in professional conferences, but more importantly, the results and methods have been utilized in the actual product developments through sponsors and effective dissemination in the industry. There is no systematic implementation of these methods in computer programs in or out of this industry primarily because of the specialized applications in a very small sector of the electronics industry while the subject has been traditionally studied by researchers in the field of solid mechanics, which is considered as one the difficult and hard subject with no direct interests and primary applications in electronics. Engineers have to go through the theoretical formulation, mathematical programming, and experimental procedure as an interdisciplinary undertaking to satisfy the needs of design process. It is understandable that details of such implementation are not publicized to outsiders, and the commercialization is not an option due to the relatively small market and heavy dependence on experiences.

In the course of the technology sophistication in the period of almost a century and gradual introduction of software tools, analytical techniques of quartz crystal resonators have been aided by many major scientific and technological breakthroughs including the Mindlin plate theory which is considered one of great achievements in improving the classical plate theory in certain applications such as the thickness-shear vibrations of crystal resonators, and the effects of bias fields on the vibrations of elastic solids with the piezoelectric solids as one of most important applications [4, 7]. Besides, many advanced technology such as finite element method and supercomputers have been utilized in the analysis of mechanical vibrations at the RF range by leading institutions through various

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This research is supported by grants from TXC (Ningbo) Corporation, the Bureau of Science and Technology, City of Ningbo under the Key Technology Initiative (Project 2007B10052) and the National Natural Science Foundation of China (Projects 10932004 & 10772087).

partnerships [6, 11, 14, 19]. These efforts have produced reliable results for resonator analysis and design, and the follow up efforts by major producers of quartz crystal resonators have shown that the traditional products and industry can be transformed by the digital technology for the revolution in the design and manufacturing processes.

Having witnessed the power and efficiency of supercomputing and finite element method for the analysis of high frequency vibrations of quartz crystal resonators and conducted extensive studies on the simplified formulation and analysis with plate theory for the same problem, the validation experiences in the past decades have indicated that the methods and solutions based on the plate theory have produced reliable results with adequate precision for product development. What is missing in the past is that the required validation procedures and results have not been reported and the user side is required to provide additional contribution. Besides, a quartz crystal resonator producer should commit to the systematic implementation and necessary collaboration between process and design engineers. Fortunately, such interests are on the rise with some emerging industrial leaders due to limited accumulation of expertise and systematic mastering of the fundamental theory of analysis. A systematic exposure to the theory for analysis of quartz crystal plates and integration of analytical methods would establish the firm ground for the design and development of products based on analysis and digital technology as emphasized as one of the technical trends in the transformation of traditional products of the manufacturing business.

In this paper, we shall review the accomplishments of extensive studies on the theory and solutions based on the high frequency vibrations of quartz crystal plates, and summarize up the available methods, software components, and verified results and data for architecting the design software for the initial design with the integration of analytical results, empirical data, and best practice in a more user friendly computer platform and convenient programming environment such as the symbolic languages.

II. THEORY AND METHODS FOR THE ANALYSIS OF QUARTZ CRYSTAL RESONATORS

The fundamental theory for the high frequency vibrations of quartz crystal plates starts from the inclusion of the thickness-shear vibration mode and free boundary conditions [1-3, 5, 8]. Of course, complication factors such as electrodes on the plate, piezoelectric effect, thickness variation, and the anisotropic material properties have also been considered in the formulation [8, 12-13]. The theory of crystal plate vibrations is known as the Mindlin plate theory with focused applications in quartz crystal resonators. The analytical results of such equations, in most cases, are given in straight-crested wave solutions which can only accommodate partial boundary conditions [1-3]. The accuracy is adequate for resonator analysis and design because many measurements have confirmed the one-dimensional results and late analysis by the finite element method shows that the deformation in the direction perpendicular the direction of wave propagation is almost uniform and the assumption is valid in the vicinity of the thickness-shear resonance.

To recapitulate the analytical methods and results, we start from the fundamental theory and some approximations to work on details of equations and some basic physical characters for analysis.

1. Mindlin plate equations for the coupled thickness-shear vibrations

It is essential to analyze high frequency vibrations, which implies the modes are different from the usual flexural vibrations discussed in most textbooks on plate theory and vibrations. The thickness-shear vibrations, which have frequency dependence on the thickness of plates, have to be studied by the Mindlin plate theory, which was established by Mindlin for the analysis of quartz crystal resonators and lately used in other problems of structural vibrations in broad engineering fields [1, 2, 9, 20-21]. The essence of the Mindlin plate theory is that displacements are expanded in power series of the thickness coordinate, thus presenting the thickness-shear displacement associated with the linear term of thickness coordinate as an independent variable. This makes it possible to analyze vibrations of finite plates because earlier methods for vibrations of thickness-shear mode can only be one-dimensional and thickness dependent. For a plate with two major faces in Fig. 1, a systematic derivation of the Mindlin plate equations starts from the displacement expansion [1, 2, 6]

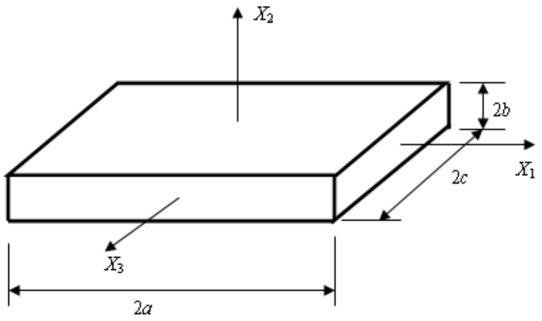

Fig.1 A typical plate with coordinate system

$$u_j(x_1,x_2,x_3,t)=\sum_{n=0}^{\infty} u_j^{(n)}(x_1,x_3,t)x_2^n, j=1,2,3, \qquad (1)$$

where $u_j, u_j^{(n)}$, x_i, and t are displacements, *n*th-order displacements, coordinates, and time, respectively. Further derivation of the plate equations is given by Mindlin [1-2].

Although the Mindlin equations have been developed for the analysis of vibrations of quartz crystal plates with finite size, the mathematical solutions satisfying the boundary conditions, with the major faces handled through the integration over the thickness already, at the length and width sides are still impossible to obtain for free edges. As a result, the effects of width on the vibrations have been neglected with the assumption that the wave mode only varies along the direction of wave propagation. This assumption has been employed without further explanation, but our recent finite element analysis has been shown that for most plates, displacement variation along the width direction is indeed small and the negligence is justifiable. Most analytical solutions based on the Mindlin plate equations with multiple modes are obtained based on the straight-crested wave assumptions. The analytical solutions are obtained with major vibration modes like the thickness-shear (TSh), flexural (FL), face-shear (FS), and electrical potentials as the framework [1, 2, 5, 8, 15].

2. *Effects of electrodes on vibrations of resonators*

Extensive studies on the effects of electrodes have never been enough to meet the needs of product development and improvements [3, 7, 12, 13, 16]. The reasons include the fact that the driving of resonator has to be enabled by electrodes and the frequency and electrical parameters are also directly affected by the material, size, method, and location of processing of electrodes. Energy trapping is also the direct effect of electrodes on a quartz crystal plate [3]. We need to study electrodes in vibration analysis, optimal shape and size, and the manufacturing techniques to obtain best performance of resonators. Indeed, all these issues have been studied in the course of resonator production, and there are important findings such as the optimal shapes and sizes and techniques on electrodes [8, 12-13]. Later studies on the calculation of precise frequency effect of electrodes will be important in the consideration of electrodes and crystal blank information for the initial selection of design parameters [12-13].

3. *Frequency-temperature relations*

The frequency-temperature relations of quartz crystal resonators are important due to the stringent requirement on the frequency stability in modern electronics. The analysis has to be made with the consideration of thermal fields in the vibrations of quartz crystal plates under the general theory of elasticity subject to bias fields [4-6, 14]. However, the high precision requirements of the analysis demand thermal elastic properties of quartz crystals, a material which is not widely used and tested, to be measured for the analytical framework. Not surprisingly, such a task has been undertaken by many researchers, and the material properties have been gathered to facilitate the establishment of nonlinear thermal equations of vibrations of quartz crystal plates. As a result, the widely known cubic relations of the thickness-shear vibration and the temperature have been proven with thermal equations, and the accuracy has also been proven through more reliable data from the finite element analysis [6, 14]. The thermal equations, as one can expect, also have been implemented with the Mindlin plate equations and the results based on the straight-crested wave solutions have been effective for the frequency-temperature solutions [4]. With considerations of the electrodes, beveling, the analysis of frequency-temperature relations has been closer to actual resonators with reliable predictions for the improvement of design.

4. *Electrical parameters of quartz crystal resonators*

As an electronic component for circuits, the usual requirements of quartz crystal resonators are specified as electrical parameters like resistance, capacitance, inductance, and quality factors. It is always expected that the electrical properties of a quartz crystal can be extracted from the vibration solutions for circuit analysis and direct applications and compared with product testing and measurement. This is possible if the mechanism related to circuit parameters is incorporated into the formulation of vibrations. It turns out that major parameters can be calculated with the consideration of material viscosity which is related to the resistance and quality factor through the energy loss in the vibrations. The material viscosity of quartz crystals is known through earlier measurements [22], and a formulation of the major parameters have been given, paving the path for the formulation of electrical parameters with the Mindlin plate theory [17]. The results show that the parameters from the theoretical calculation with material viscosity are much smaller than the actual measurements, and it is suspected that the viscosity from the bonding process of the electrodes and other factors are not considered but significant.

III. VERIFICATION OF ANALYTICAL METHODS

There have been tremendous efforts on the verification of results while the analysis with approximate equations and simplified models are being used for the design and optimization of quartz crystal resonators. The general verification of the Mindlin plate theory has been done with the accurate solutions from the three-dimensional equations of wave propagation in elastic solids with the conclusion that the predictions with properly selected modes should be adequately precise in the vicinity of thickness-shear vibrations. Even with these simple objectives, there are many testing and measurement parameters and we can select a few from the important ones and also the ones which can be easily measured without going through a process to isolate the cross-talks between them.

1. *The precise thickness of crystal blanks and electrodes*

It is known that the crystal blanks and electrodes are processed by not controlling the exact thickness but the exact frequency through continuous measurements. Eventually, the processing is controlled through time after the empirical knowledge is established. As a result, the exact thickness of the crystal blank and electrodes may not necessarily the most important parameters in the design process because they usually do not come into the calculation directly. As we gradually adopt the calculation-based design process, this will change inevitably because the thicknesses will appear in more calculations eventually like the electrical circuit parameters. For this reason, we need to establish the relationship between the crystal blank, electrodes and the frequency in a resonator. Fortunately, the relationship has been given by Tiersten and Bleustein [23] and later polished by Wang and Shen [13]. Using the known equations for the thickness-shear vibrations of an infinite plate with the consideration of piezoelectric effect, we have the measured and calculated frequencies with electrodes plated at the TXC Ningbo Plant given in the Table 1. We can see that the calculated results and measurements agree well for a fundamental mode AT-cut resonator at about 40MHz, and it is considered as the start of the design process.

Table 1 Measured and calculated thickness of electrode crystal blanks

Measured (μm)	40.3	39.7	39.1	39.3	39.8	40.0	39.3
Calculated (μm)	40.0	39.6	39.2	38.8	39.6	39.6	39.6

2. *The frequency and effect of face-shear vibration modes*

It is generally known that there are many unwanted, or spurious, modes in a quartz crystal resonator, and the design and improvement is always on the suppression, or avoidance, of these modes to obtain stable frequency output in the operating temperature range. The functioning mode is the thickness-shear, and the mostly encountered spurious modes are the flexural and face-shear (thickness-twist sometimes) modes, and they can be spotted from the measurement of the frequency-temperature curve. To indentify the vibration modes interrupt the frequency behavior, we calculate the corresponding frequencies of different modes and make comparisons to label them for possible suppression. In this study, we use again the 40 MHz AT-cut resonator with blanks of 2.0 × {1.295, 1.315, 1.335,1.355} mm and calculated vibration frequencies of the face-hear (or the thickness-twist) mode with the simplified equations from Mindlin [24-25]. Surprisingly, the identified face-shear modes from the frequency of couplings agree well with the calculations based on the simple equations of Mindlin as shown in Fig. 2. Further comparison with the plate equations on the face-shear modes is being made.

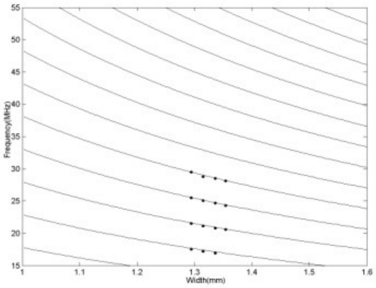

Fig. 2 Calculated and measured frequencies of the face-shear mode of a quartz crystal resonator

The available analytical methods and validation procedures above will be the foundation for the design and implementation of design software of quartz crystal resonators with the participation of TXC engineers.

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