Parameters Extraction and Design Optimization for AT-cut Quartz Resonator based on Mindlin's 2D Model

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***Abstract*—For miniaturized strip AT-cut quartz resonator, the demand of performance is higher and higher, so the design ability is more important than before. But to get a precise simulation electrical response of the AT-cut quartz plate by a common commercial FEM tool is not easy and time consuming also. Base on Mindlin's [1] 2-D model and Lee-Brebbia's [2] FEA method, Pao [3] et al. presented an efficient numerical method in calculating the electrical response different modes of AT-cut strip quartz crystal resonator with electrode. Base on weak coupling of quartz material and mass loading effect of electrode, we solve this problem by separating the mechanical vibration and electrical coupling effects. The method considers not only the pure mechanical vibration but also the electrical response, so we can identify different modes effectively and efficiently. However, to be a practical design tool, the parameters used in the simulation is better come from real samples.**

In this paper, we extend the works to parameters extraction for design optimization. A blank with electrode was built and its acoustic and electrical material parameters (including damping coefficient) of the quartz resonators could be extracted by fitting to few measured sample admittance curves. The extracted parameters could be used for design optimization purpose. To exam the effectiveness of the parameters extraction, another blank with electrode quartz resonator was built and the results showed the calculated impedance curves could match the measured ones, and a practical AT-cut strip quartz resonator design tool could be made based on extended Minlin's 2D model.

Keywords- AT-cut quartz resonator; electrrical impedance; parameter extraction

I. INTRODUCTION

In order to develop an efficient and precise enough simulation tool for miniature AT-cut quartz resonator for high performance application, we use a two dimension finite element model (base on Mindlin-Lee's model) to simulate the electrical response of the AT-cut quartz plate with electrode. In our prior work, we had developed a finite element tool which can not only analyze the pure mechanical vibration but also the electrical impedance response. However, because it is hard to measure or calculate the acoustic or electrical impedance, we did not fit the simulation results and experiment response before. To solve this problem, we extract the parameters from the AT-cut quartz resonator sample, and then fine tune the input coefficients in the simulation tool, e.g. damping coefficient, dielectric constants, and stiffness constants, etc., to fit the simulation results to the extract parameters from the experiment.

In this paper, first we follow Mindlin-Lee's two dimension model to build a nine nodes quadratic finite element model, then we explain how we extend Lee's pure mechanical vibration analysis to the piezoelectric effect of the quartz material to obtain the electrical impedance response under applied electrical potential. Finally, we show how we extract the parameters of the AT-cut quartz resonator sample and fine tune the input coefficients of this design tool to fit the simulation results and the measurement response.

II. THEORY OF THE QUARTZ PLATE VIBRATION

According to Mindlin's two-dimensional plate equations [1] and neglecting the coupling with extensional motions by setting $\overline{c}_{14} = \overline{c}_{34} = c_{56} = 0$, we obtain [2]:

$$Q_{1,1} + Q_{3,3} = 2b\rho\ddot{u}_2 \tag{1a}$$

$$\mathrm{M}_{1,1} + \mathrm{M}_{5,3} - Q_1 = \frac{2}{3}b^3\rho\ddot{\psi}_1 \tag{1b}$$

$$\mathrm{M}_{5,1} + \mathrm{M}_{3,3} - Q_3 = \frac{2}{3}b^3\rho\ddot{\psi}_3 \tag{1c}$$

where M_1 and M_3 are bending moment; M_5 is twist moment; Q is shear force; b is half thickness; ρ is density of quartz; ψ_1 and ψ_3 are the rotating angle along axis x_3 and x_1.

For strip quartz plate, we use the nine-node quadratic element as the finite element analysis elements. The element dimensions are W and L along x and y directions, respectively, and the interpolation functions are [4]:

$$\varphi = \begin{bmatrix} \varphi_1 & \varphi_4 & \varphi_7 \\ \varphi_2 & \varphi_5 & \varphi_8 \\ \varphi_3 & \varphi_6 & \varphi_9 \end{bmatrix} \tag{2}$$

where

$$\varphi_1 = \left(1 - \frac{2x}{W}\right)\left(1 - \frac{x}{W}\right)\left(1 - \frac{2y}{L}\right)\left(1 - \frac{y}{L}\right)$$

$$\varphi_2 = \frac{4x}{W}\left(1-\frac{x}{W}\right)\left(1-\frac{2y}{L}\right)\left(1-\frac{y}{L}\right)$$

$$\varphi_3 = \frac{x}{W}\left(\frac{2x}{W}-1\right)\left(1-\frac{x}{W}\right)\left(1-\frac{2y}{L}\right)\left(1-\frac{y}{L}\right) \quad (2a)$$

$$\varphi_4 = \left(1-\frac{2x}{W}\right)\left(1-\frac{x}{W}\right)\frac{4y}{L}\left(1-\frac{y}{L}\right) \qquad \varphi_5 = \frac{4x}{W}\left(1-\frac{x}{W}\right)\frac{4y}{L}\left(1-\frac{y}{L}\right)$$

$$\varphi_6 = \frac{x}{W}\left(\frac{2x}{W}-1\right)\frac{4y}{L}\left(1-\frac{y}{L}\right) \qquad \varphi_7 = \left(1-\frac{2x}{W}\right)\left(1-\frac{x}{W}\right)\frac{y}{L}\left(\frac{2y}{L}-1\right)$$

$$\varphi_8 = \frac{4x}{W}\left(1-\frac{x}{W}\right)\frac{y}{L}\left(\frac{2y}{L}-1\right) \qquad \varphi_9 = \frac{x}{W}\left(\frac{2x}{W}-1\right)\frac{y}{L}\left(\frac{2y}{L}-1\right)$$

Based on the Mindlin's 2D model, there are three degrees of freedom for each node in the element (one displacement and two rotating angles), so the interpolation function of one element is:

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}_{3\times 27} \quad (3)$$

where

$$\phi_1 = \begin{bmatrix} \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 & 0 & 0 & \cdots & \varphi_9 & 0 & 0 \end{bmatrix}_{1\times 27}$$

$$\phi_2 = \begin{bmatrix} 0 & \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 & 0 & \cdots & 0 & \varphi_9 & 0 \end{bmatrix}_{1\times 27} \quad (3a)$$

$$\phi_2 = \begin{bmatrix} 0 & 0 & \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 & \cdots & 0 & 0 & \varphi_9 \end{bmatrix}_{1\times 27}$$

By the finite element displacement method for Mindlin's 2D model developed by P.C.Y. Lee et al. [2], the equation of motion for an element (pure mechanical vibration, without damping):

$$[M^e]\{\ddot{U}^e\} + [K^e]\{U^e\} = \{0\} \quad (4)$$

where

$\left[U^e\right]$ is the displacement of the element

$$\left[M^e\right] = \int_A [\phi]^T [m][\phi] dA$$

$[m]$ is the mass ratio matrix of the element (4a)

$$\left[K^e\right] = \int_A \left([B_b]^T [D_b][B_b] + [B_t]^T [D_t][B_t]\right) dA$$

$$[B_b] = \begin{bmatrix} \phi_{2,1} \\ \phi_{3,3} \\ \phi_{3,1} + \phi_{2,3} \end{bmatrix}_{3\times 27} \qquad [B_t] = \begin{bmatrix} \phi_{1,1} + \phi_2 \\ \phi_{1,3} + \phi_3 \end{bmatrix}_{2\times 27}$$

$[D_t]$, $[D_b]$ are the stiffness matrix composed by the stiffness constants c_{ij}, developed in [2]

For the piezoelectric effect, we need to extend the pure mechanical equations of motion to the piezoelectric term [5].

$$\begin{bmatrix} M^e & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{U}^e \\ \ddot{V}^e \end{Bmatrix} + \begin{bmatrix} K^e & K^e_{uV} \\ K^{e\ T}_{uV} & K^e_{VV} \end{bmatrix} \begin{Bmatrix} U^e \\ V^e \end{Bmatrix} = \begin{Bmatrix} F^e \\ Q^e \end{Bmatrix} \quad (5)$$

where F^e, Q^e, and V^e are force, charge, and potential on the element respectively. For the Mindlin's two dimension plate vibration theory, the matrices connect mechanics and electric are:

$$\left[K_{u\phi}\right] = \int_A \left(\left[B_u\right]^T [e]^T \left[B_\phi\right]\right) dA \quad (6)$$

$$\left[K_{\phi\phi}\right] = \int_A \left(\left[B_\phi\right]^T [\varepsilon]^T \left[B_\phi\right]\right) dA \quad (7)$$

where

$$[B_u] = \begin{bmatrix} \phi_{3,3} \\ \phi_{2,1} \\ \phi_{1,1} + \phi_2 \\ \phi_{1,3} + \phi_3 \\ \phi_{2,3} + \phi_{3,1} \end{bmatrix} \text{ and } [B_u] = \begin{bmatrix} \phi_{4,1} \\ \phi_{4,2} \\ \phi_{4,3} \end{bmatrix} \quad (8)$$

III. Solving the Equations of Motion with Piezoelectric Effect

For the calculation efficiency, based on weak piezoelectric coupling, we solve the mechanical and electrical potential parts separately. First we solve the pure mechanical vibration as an eigenmode problem.

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \quad (9)$$

Solving (9), we can get the eiengmodes (no external force), $v_1, v_2, v_3, \cdots v_n$, and the eigenvalues, $\omega_1, \omega_2, \omega_3, \cdots \omega_n$. Finishing the pure mechanical vibration analysis, we can get only natural frequencies and shapes in free vibration, but no response amplitude. Many numerical modes even do not exist in the measured results.

To get the electrical potential, we rearrange the electrical potential applied on the quartz plate in (5) to the right hand side of the elastic-piezoelectric equations of motion. These terms become the forcing terms of the pure elastic equations of motion:

$$[K_{uV}]\{V(\omega)\} \to \{F_e(\omega)\} \quad (10)$$

in (10), we consider the applied electrical potential V is sinusoidal $V(\omega)$; the subscript e in F means the applied external force by electrical potential. Add the forcing and damping term into the equations of motion of the vibration plate:

$$[M]\{\ddot{u}\} + [\eta]\{\ddot{u}\} + [K]\{u\} = \{F_e(\omega)\} \quad (11)$$

here we consider the damping is classical damping, there are only values in the diagonal. Using the mode superposition method [6] to solve u:

$$u = \alpha_i v_i + \alpha_{i+1} v_{i+1} + \alpha_{i+2} v_{i+2} + \cdots + \alpha_k v_k \tag{12}$$

where $v_i, v_{i+1}, v_{i+2}, \cdots, v_k$ are the free vibration eigenmodes we choose to superpose; and $\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \cdots, \alpha_k$ are weights.

We use eigen-expansion to decouple (11) to get decoupled ODEs for the weight of each eigenmode. The elastic displacement field, u, with electrical potential effect can then be calculated as the weighting sum of each eigenmode. The electrical charges on the quartz crystal resonator can be obtained from the elastic displacement and electrical potential field at each node point:

$$\begin{bmatrix} K_{uV}{}^{T} & K_{VV} \end{bmatrix} \begin{Bmatrix} u \\ V \end{Bmatrix} = \{Q(\omega)\} \tag{13}$$

and then we can get the electrical response, admittance Y:

$$\frac{\left(\sum Q_i\right) \cdot f}{V} = Y \tag{14}$$

IV. PARAMETER EXTRACTION

The equivalent BVD circuit elements, R_m, C_m, L_m, and C_0, of a resonator can be extracted by the admittance circle obtained by measuring the resonator sample.

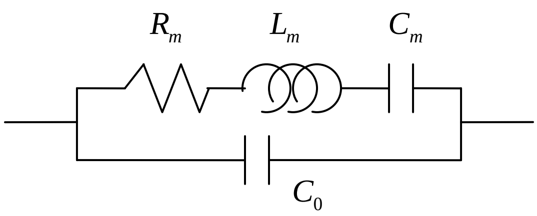

Figure 1. The equivalent circuit of a resonator

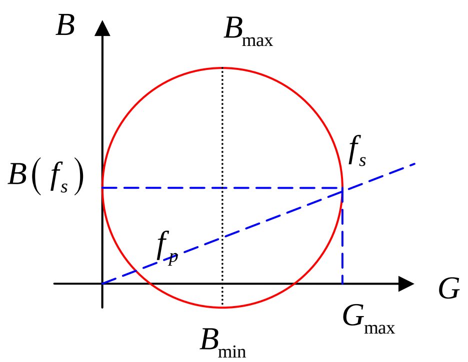

Figure 2. The admittance circle of a resonator

We can obtain the equivalent series resistant, R_m, by inversing G_{max} obtained from the admittance circle.

$$R_m = 1/G_{max} \tag{15}$$

and then we can calculate the parallel capacitance C_0, motional capacitance C_m, and motional inductance L_m by the following equations:

$$Y(f_s) = G_{max} + jB(f_s) = G_{max} + j \cdot 2\pi f_s \cdot C_0$$

$$B(f_s) = (B_{max} + B_{min})/2 \tag{16}$$

$$C_m / C_0 = (f_p^2 - f_s^2) / f_s^2 \tag{17}$$

$$f_s = 1 / \left(2\pi \sqrt{C_m \cdot L_m}\right) \tag{18}$$

Figure 3 is the admittance circle by measuring a AT-cut quartz resonator sample.

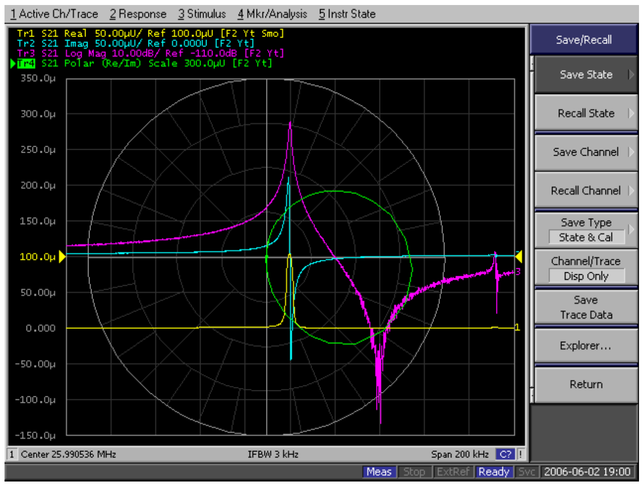

Figure 3. The admittance circle of a AT-cut quartz resonator sample. The yellow line is the real part of the admittance; the indigo line is the image part of the admittance; the pink line is the logarithm scale magnitude of the admittance; and the green line is the admittance circle.

By the measurement result in figure 3, we can obtain the equivalent circuit elements:

$$R_m = 13.7\Omega$$

$$C_0 = 1.5\,pF$$

$$C_m = 2.1\,fF$$

$$L_m = 5.2\,mH$$

after getting these parameters, we fine tune the input coefficients in our finite element design tool to fit the measurement result, as figure 4 shows.

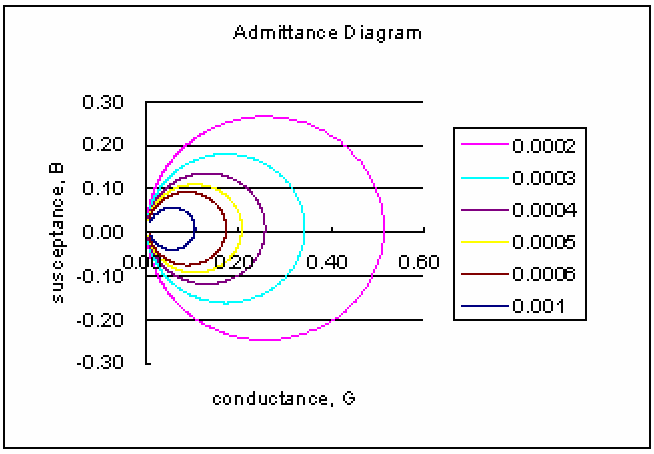

Figure 4. Changing the input damping coefficient of the finite element design tool to fit the measurement results of the sample

Figure 5 shows that after we tune the input parameters of the design tool appropriately, we can fit the admittance response of the simulation results and measurement data very well.

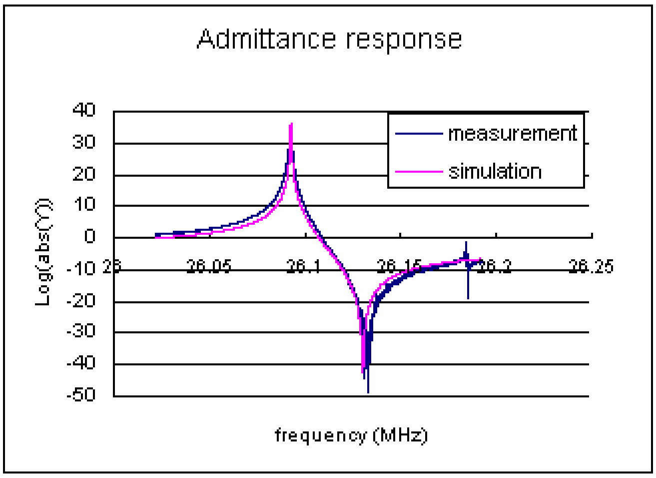

Figure 5. The comparison of the fitting curve of the simulation and the experiment measurement data

V. Conclusion

Following our prior work, based on Mindlin 2-D elastic-piezoelectric model and weak piezoelectric coupling assumption, we develop an efficient numerical method to analyze the AT-cut quartz resonator. The acoustic and electrical material parameters (including damping coefficient, stiffness constants, and dielectric constants) of the quartz resonators could be extracted by fitting to a few measured sample admittance curves. The extracted parameters could be used for design optimization purpose. Using these extracted parameters, we decide the input coefficient of the design tool for any other resonator analysis to obtain the precise electrical response.

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